Quantum calculus and singularities of quasi-discriminant sets

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Let $g:\mathbb{R}\to\mathbb{R}: x\mapsto g(x)$ be a given smooth one-to-one map of the real axis, which is the domain of polynomial $f(x)$ with arbitrary coefficients. We want to find conditions on the coefficients of the polynomial under which it has at least a pair of roots t_i, t_j satisfying the relation $g(t_i)=t_j$ and investigate the structure of the algebraic variety in the space of coefficients possessing such property.

Here we consider a generalization of the classical discriminant of the polynomial. This generalization naturally includes the classical discriminant and its analogs emerging when the q-differential and difference operators are used. The aim of this research is to propose an efficient algorithm for calculating the parametric representation of all components of the g-discriminant set $\mathcal{D}_g(f)$ of the monic polynomial $f_n(x)$ of degree n.

Define the q**-bracket** $[a]_q$, q**-Pochhammer symbol** $(a;q)_n$, q**-factorial** $[n]_q!$, q**-binomial coefficients (Gaussian) coefficients** $\left[{n \atop k}\right]_q$, g**-binomial** $\{x;t\}_{n;g}$ as follows: $[a]_q=\dfrac{q^a-1}{q-1}$, $(a;q)_n=\prod\limits_{k=0}^{n-1}\left(1-aq^k\right)$, $(a;q)_0=1$, $[n]_q!=\prod\limits_{k=1}^{n}[k]_q=\dfrac{(q;q)_n}{(1-q)^n}$, $q\neq 1$, $\left[{n \atop k}\right]_q=\frac{[n]_q!}{[n-k]_q!\,[k]_q!}=\prod\limits_{i=1}^{k}\frac{q^{n-i+1}-1}{q^i-1}$, $\{x;t\}_{n;g}\equiv\prod\limits_{i=0}^{n-1}(x-g^i(t))$, $\{x;t\}_{0;q}=1$. Here g^k is the k-th iteration of the diffeomorphism g, $k\in\mathbb{Z}$. As $q\to 1$, all these objects become classical.

Let $f_n(x)$ be is a monic polynomial of degree n with complex coefficients defined by $f_n(x)\stackrel{\text{def}}{=}x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n$. Let $\mathbb{P}$ be the space of polynomials over $\mathbb{R}$ and let $g:\mathbb{R}\to\mathbb{R}:x\mapsto qx+\omega$, $q,\omega\in\mathbb{R}$, $q\neq\{-1,0\}$, be a linear diffeomorphism on $\mathbb{R}$ that induces a linear **Hahn operator** $\mathcal{A}_g$ on $\mathbb{P}$, satisfying the following two conditions: (1) the degree reduction: $\deg(\mathcal{A}_g f_n)(x)=n-1$; in particular, $\mathcal{A}_g x=1$; (2) Leibnitz rule analogue: $(\mathcal{A}_g x f_n)(x)=f_n(x)+g(x)(\mathcal{A}_g f_n)(x)$.

The Hahn operator $\mathcal{A}_g$ called below g**-derivative** has the form

$$(\mathcal{A}_g f)(x)\stackrel{\text{def}}{=}\begin{cases}\dfrac{f(qx+\omega)-f(x)}{(q-1)x+\omega}, & x\neq\omega_0,\\ f'\left(\omega_0\right), & x=\omega_0,\end{cases}$$

where $\omega_0=\omega/(1-q)$ is the fixed point of g. Parameters q and ω are satisfied the following conditions $q,\omega\in\mathbb{R}$, $q\neq\{-1,0\}$ and $(q,\omega)\neq(1,0)$. The g-derivative $\mathcal{A}_g$ can be considered as a generalization of the q-differential Jackson operator $\mathcal{A}_q$ at $\omega=0$, $q\neq 1$, as the difference operator Δ_ω at $q=1$ and as the classical derivative d/dx in the limit $q\to 1$ and $\omega=0$.

The q-calculus has became a part of the more general construct called quantum calculus [1, 2]. It has numerous applications in various fields of modern mathematics and theoretical physics. The pair of roots t_i, t_j, $i,j=1,\dots,n$, $i\neq j$ of the polynomial $f_n(x)$ is called g**-coupled** if $g(t_i)=t_j$.

Problem 1. In the coefficient space $\Pi\equiv\mathbb{C}^n$ of the polynomial $f_n(x)$, investigate the g**-discriminant set** denoted $\mathcal{D}_g(f_n)$ on which this polynomial has at least one pair of g-coupled roots.

The sequence $\mathrm{Seq}_g^{(k)}(t_1)$ of g-coupled roots of length k is defined as the finite sequence $\{t_i\}$, $i=1,\dots,k$ in which each term, beginning with the second one, is a g-coupled root of the preceding term: $g(t_i)=t_{i+1}$. The initial root t_1 is called the generating root of the sequence $\mathrm{Seq}_g^{(k)}(t_1)$.

For each fixed set of parameters q, ω, the g-discriminant set $\mathcal{D}_g(f_n)$ consists of a finite set of varieties $\mathcal{V}_k$ on each of which $f_n(x)$ has k sequences $\mathrm{Seq}_g^{(l_i)}(t_i)$ of g-coupled roots of length l_i with different generating roots t_i, $i = 1, \ldots, k$. To obtain an expression for the generalized (sub)discriminant of the polynomial $f_n(x)$ in terms of its coefficients, any method available in the classical elimination theory can be used. If we replace the derivative $f_n'(x)$ by the polynomial $\mathcal{A}_g f_n(x)$, then any matrix method for calculating the resultant of a pair of polynomials gives an expression of the generalized k-th subdiscriminant $D_g^{(k)}(f_n)$ [3].

Theorem 2. *The polynomial $f_n(x)$ has exactly $n-d$ different sequences of g-coupled roots, iff the first nonzero element in the sequence of i-th generalized subdiscriminants $D_g^{(i)}(f_n)$ is the subdiscriminat $D_g^{(d)}(f_n)$ with the index d.*

Consider the partition $\lambda = [1^{n_1} 2^{n_2} 3^{n_3} \ldots]$ of a natural number n. Every partition λ of n determines the structure of the g-coupled roots of the polynomial $f_n(x)$, and this structure is associated with the algebraic variety $\mathcal{V}_l^i$, $i = 1, \ldots, p_l(n)$ of dimension l corresponding to the number of different generating roots t_i in the coefficient space Π. The partition $\left[n^1\right]$ corresponding to the case when there is a unique sequence of roots of length n specified by the generating root t_1. Then, the polynomial $f_n(x)$ is a g-binomial $\{x; t_1\}_{n;g}$ and its coefficients a_i can be represented in terms of the elementary symmetric polynomials $\sigma_i(x_1, x_2, \ldots, x_n)$ calculated on the roots $g^j(t_1)$, $j = 0, \ldots, n-1$, $a_i = (-1)^i \sigma_i\left(t_1, g(t_1), \ldots, g^{n-1}(t_1)\right)$, $i = 1, \ldots, n$.

Theorem 3 ([4]). *Let there be a variety $\mathcal{V}_l$, $\dim \mathcal{V}_l = l$ on which the polynomial $f_n(x)$ has different sequences of g-coupled roots and the sequence of roots $\mathrm{Seq}_g^{(m)}(t_1)$ has length $m > 1$. The roots of the other sequences are not g-coupled with all roots of the sequence $\mathrm{Seq}_g^{(m)}(t_1)$. Let $\mathbf{r}_l(t_1, \ldots, t_l)$ be a parameterization of the variety $\mathcal{V}_l$. Then for $0 < k < n$, the formula*

$$\mathbf{r}_l(t_1, \ldots, t_l, t_{l+1}) = \mathbf{r}_l(t_1, \ldots, t_l) + \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{[m-i]_q!}{[m]_q!} \left(\mathcal{A}_g^i \mathbf{r}_l\right)(t_1) \{t_{l+1}; t_1\}_{i;g} \tag{1}$$

specifies a polynomial parameterization of the part of $\mathcal{V}_{l+1}$ on which there are two sequences of roots $\mathrm{Seq}_g^{(m-k)}(g^k(t_1))$ and $\mathrm{Seq}_g^{(k)}(g(t_{l+1}))$, and the other sequences of roots are the same as on the original variety $\mathcal{V}_l$.

The structure of singular points of each variety $\mathcal{V}_{l+1}$ can be described in terms of varieties $\mathcal{V}_l$, connected with it by (1).

The same results on the structure and parametrization of the g-discriminant set $\mathcal{D}_g(f_n)$ can be obtained for other variants of g-derivative, e.g. for the case of Hahn symmetric derivative [2]

$$\widetilde{\mathcal{A}}_{q,\omega} f(t) = \frac{f(qt+\omega) - f(q^{-1}(t-\omega))}{(q-q^{-1})t + (1+q^{-1})\omega}$$

as well.

References

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